

INSTABILITY OF THE ICE FREE EARTH: DYNAMICS OF A DISCRETE TIME ENERGY BALANCE MODEL *

ESTHER R. WIDIASIH

Abstract. In late sixties, Mihail Budyko and William Sellers, a Russian and an American climate scientists, independently introduced the concept of Energy Balance Model with ice albedo feedback. Since then many have followed in their footsteps to establish various versions of this model. In this paper, a novel equation is introduced to account for the dynamics of the ice line, and is coupled to Budyko's model. We found that the coupled temperature profile-ice line system has a one dimensional center stable manifold. Furthermore, this conceptual model, that accounts only for feedback from the ice albedo and that should not be used for predictive purposes, suggests that under some climate forcings, the ice free earth is unstable.

Key words. invariant manifolds, stability, energy balance model, infinite dimensional dynamical systems, graph transform method, climate, ice albedo feedback, conceptual model.

AMS subject classifications. Preprint

1. Introduction. Energy Balance Models (EBMs) rest on the concept that in the earth's equilibrium state the energy received from the sun's radiation is balanced by the energy re-emitted back to space at the earth's temperature. Budyko's EBM aims to model the latitudinal distribution of the surface annual mean temperature, by taking into account the feedback from ice albedo (icecap's reflectivity factor). As a dynamical system then, Budyko's model takes place in an infinite dimensional state space. While the discussion of climate feedbacks easily enters realm of great complexity, the concept of ice albedo feedback and its effect on the planet, is not difficult to explain. Ice albedo is the degree of which ice reflects light or energy. Since ice cover in general has whiter color than the blue ocean, it reflects more light or energy back into space. This creates a positive feedback, properly called the *ice albedo feedback*, because more ice means less energy absorbed, which induces cooler climate favoring condition for more ice formation, and so on. On the other hand, shrinkage of ice mass induces more ocean surface, which absorbs more energy, creates warmer climate and in turn, promotes more melting. But despite this easy to explain concept, the conclusions on the stability of icecaps vary greatly, even among the simplest models. While previous results were useful in understanding steady states of the energy balance models, they did not include mechanisms for ice-water boundary or iceline movement, making it difficult to obtain rigorous results for icecap stability. In this paper, to go along with Budyko's model, we introduce an equation that describes the movement of the ice line. We call this coupled system, Dynamic Iceline Budyko's Model (DIBM), and it consists of two equations: the first is a version of Budyko's model, similar to that written by Ka Kit Tung [8] describing the evolution of the temperature profile, and the second is the novel equation that models the iceline dynamics.

Our planet currently has a small ice cover surrounding both poles, and in light of the ice albedo feedback concept, many has pondered on the stability of these polar ice caps. The idea of small ice cap instability, or sometimes abbreviated SICI,

* The author thanks Prof. McGehee, University of Minnesota, Prof. Kevin Lin University of Arizona, for their guidances. Part of this work was completed at the Center of Research in Mathematics, Montreal, Canada.

perhaps was started in 1924 by CEP Brooks [14], [13], when he hypothesized that in this planet, because of a mechanism later coined as ice albedo feedback, only two stable climates are possible: ice free and large ice capped earth, with the cryospheres extending from the poles pass the 78° latitude. In late sixties, a Russian climatologist Mihail Budyko proposed an ice albedo feedback model based on the energy balance principle, where transport is represented by a simple relaxation process in which the temperature of each latitude dissipates to the global average temperature. He concluded that both ice free and polar or small ice cap climates are unstable (see p. 618 [7]). Around the same time, William Sellers from University of Arizona proposed a minimal complexity climate model which includes planetary albedo, but uses separate atmospheric and oceanic transport processes [12]. In the next decade, Gerald North explored in depth the diffusive transport version of the energy balance model with the ice albedo feedback, and in his works, upon varying some of the parameters, the small ice cap instability disappears [2], [3].

While the small ice cap instability discussions in the mid twentieth century were fuelled by the possibility that the planet is going into its glaciation period, the more recent discussion on the state of the cryosphere is motivated by the rapid decrease of the Arctic sea ice extent, induced by global warming and placed more emphasis on the reversibility of an ice free state in a warmer climate. Recent results that attempt to answer this question have also been characterized by the use of computer simulations such as the one developed by in 2008 by Merryfield, Holland and Monahan [11], a simple climate model based on the Community Climate System Model version 3 (CCSM 3)-a global climate model developed by NCAR. The simple model is nonlinear and admits abrupt sea ice transitions resembling those in the CCSM 3 simulations. In early 2009, Eisenman and Wettlaufer [1] examined an energy balance model with seasonal features and a nonlinear forcing given by the sea ice thermodynamics. Their results suggest that there exists a critical threshold in the climate warming, beyond which a sudden loss of the remaining seasonal sea ice is possible. Another recent paper published in 2009 by Notz [13] summarized the discussions on the state of the "existence of the cryospheric tipping points".

We will show that Dynamic Iceline Budyko's Model admits an unstable large ice cap, a stable ice covered planet, a stable small ice cap, and an unstable ice free planet indicating polar ice cap loss reversibility. Furthermore, we show that the infinite dimensional system has a 1-D center stable manifold, hence, the dynamic is essentially one dimensional. The existence of this one dimensional attractor will also be illustrated in the computer simulations, but in the end, it is the mathematical analysis of the dynamics that gives us the confidence in the numerical executions.

Unlike the more complex computer simulation models, our result should not be seen as predictive, but instead, as a tool to understand qualitatively the mechanism involved in climate processes. The advantage of our formulation is that its tractability allows us to isolate the effects of a single climate feedback, ie, the ice albedo feedback. For example, we see from this result that despite being a positive feedback, the ice albedo feedback is not enough to cause a tipping point phenomena as seen in recent studies, see [1] and reference therein. The comparison of these two results indicates the fruitfulness of exchanges between computer simulation models and theoretical analysis, hence, between climate scientists and mathematicians.

We will organize this paper as follows: in section 2, we introduce Budyko's energy balance model along with a new feature to account for the iceline dynamics, and we discuss the equilibria of the systems. Then we show some animations that results from the numerics to illustrate the dynamics. Section 3 discusses the analysis of the Dynamics Iceline Budyko's model and its map, with detailed proofs using graph transform method in Section 4. Section 5 provides concluding remarks and a sketch of some future directions.

2. Dynamic Iceline Budyko's Model. The equation governing the temperature profile evolutions by itself does not induce the movement of the iceline, as illustrated by computer simulations in the following section. To capture the feedback from the ice albedo, we propose a system of two time scales, integro-difference equations governing the temperature distribution and the iceline dynamics. With this proposed system, we asked the following questions:

- What is the appropriate function space for the temperature profiles?
- What are the dynamics of the proposed system?
- What are the parametric conditions to have an equilibrium state at the current ice line?
- What does the model suggest if the current condition is perturbed so that the planet is ice free?

The main results of this study are:

- The identification of the function space in which the proposed system is well defined.
- The existence of an invariant manifold for the Euler approximation of the proposed system.
- A parametric condition to have an equilibrium at the present climate.
- The instability of the ice free earth.

We first introduce the details and some background of the model, then we explore the equilibria, the invariant manifold and, finally, the instability of the ice free planet.

2.1. Details of Budyko's Model. While Budyko introduced his original model as a concept, following his steps many have formulated and parametrized the energy balance and ice albedo feedback concept. KK Tung has summarized the formulation as a differential equation with a nice exposition in his book Tung (2007):

$$R \frac{\partial}{\partial t} T(y) = Q \cdot s(y) \cdot [1 - \alpha(y)] - [A + BT(y)] - C \cdot [T(y) - \int_0^1 T(y) dy] \quad (2.1)$$

The function $T(y)$ is the annual average surface temperature at the zone y , and its graph over the domain of y is called *the temperature profile*. Other previous authors, eg. [2], [10], also treated the model as a *differential equation*, hence, presupposed the continuous dependence of the annual average temperature profile $T(y)(t)$ on the time variable t . We argue that the evolution of the yearly averaged temperature profile T should be governed by a *difference equation*.

Let $\Delta_t[f](k)$ denotes the difference equation of f at time node k , that is

$$\Delta_t[f](k) = \frac{f(k+t) - f(k)}{t}.$$

We consider the following Budyko's difference equation at each time node k :

$$R\Delta_t[T(y)](k) = Qs(y)(1 - a(\eta)(y)) - (B + C)T(y)(k) - A + C\bar{T}(k) \quad (2.2)$$

Here, the constant R measures the heat capacity of the surface, and we assume that $R = 1$. Such assumption does not change the qualitative behavior of the system. The independent variable y is $\sin(\theta)$, where θ represents some latitude in the northern hemisphere, therefore y lies on the unit interval. This model assumes that the annual distribution of the surface temperature is symmetric about the equator.

We will now examine in details the terms on the right hand side of the EBM equation above. The constant Q is the solar constant, representing the amount of energy received from the sun at the top of the atmosphere. The function $s(y)$ is the latitudinal distribution of that energy which could be computed from earths orbital elements. Many authors, such as KK Tung [8], and North [2] used the Legendre polynomial function approximation $s(y) = 1 - 0.482\frac{3y^2-1}{2}$ for this distribution. We will use the same approximation as well.

The term $1 - \alpha(y)$ represents the fraction of the radiative energy absorbed by the earth at location y . To establish the ice albedo feedback effect, it is assumed that ice is formed when the temperature at a certain location stays below a critical temperature T_c , taken to be $-10^\circ C$. Also, it is assumed that the surface is either water (ice free) or ice covered and that there is only one ice line, η . Since ice reflects more sunlight than water does, area covered with ice has higher albedo than that covered with water. In this paper we use the approach that the albedo function $a(y)$ at location y depends on y relative to the location of the iceline, and not on the temperature $T(y)$. Let $\eta \in [0, 1]$ denote the location of the ice line. The albedo function we use in this paper is smooth and is iceline dependent:

DEFINITION 2.1. *Iceline-Dependent Smooth Albedo*
Given that the ice line is at η , the albedo at y is

$$a(\eta)(y) = 0.47 + 0.15 \cdot \tanh[M \cdot (y - \eta)]$$

Here M is the parameter representing the steepness of the albedo near the iceline and is a fixed quantity, presumably dictated by the planet.

As in Tung [8], balancing the absorbed radiative energy contained in the first term are the re-emission term, $A + BT(y)$ and the transport term, $C \cdot (T(y) - \bar{T})$, where \bar{T} is the global average $\int_{[0,1]} T(y)dy$. The constants $A = 202$ watts per squared meters

and $B = 1.9$ watts per squared meters per $^{\circ}C$ are derived from fitting a linear function through satellite data of Outgoing Long-wave Radiation (OLR) at the top of the atmosphere [6]. The constant C in the transport term is taken to be $1.6B = 3.04$ and is chosen so that one equilibrium fits the current climate with ice line near the pole.

2.2. Dynamic Iceline. So far we have an equation that describes the evolution of the temperature profile over the northern hemisphere, and that takes into account the ice albedo feedback. We will add to this an equation that describes the evolution of the ice line η . Previous literatures such as Budyko [7], Tung [8], North [2], etc. have used the idea that ice is formed when the temperature is below a critical threshold, T_c . We adapt this idea to the ice line evolution by prescribing a poleward movement of the ice line when the temperature at the ice line $T(\eta)$ is above a critical temperature T_c , and equatorward movement otherwise. Also, we assume that the movement of the ice line happens at a much slower rate compared to the evolution of the temperature profile. Therefore, for $\epsilon \ll 1$ and with $T_c = -10$ as used by previous authors, the equation for the ice line evolution at time node k can be written as:

$$\Delta_t[\eta](k) = \frac{\eta(k+t) - \eta(k)}{t} = \epsilon \cdot [T(\eta)(k) - T_c] \quad (2.3)$$

Some recent computation has suggested that to be physically relevant, the value of ϵ is very small, possibly in the order of $10^{(-12)}$ [22]. We will see in the simulation that using a much larger $\epsilon \cong 0.025$ the attracting 1-D manifold still appears.

The object of our analysis, Dynamic Iceline Budyko's Model, is therefore the following infinite dimensional, two time scale, integro-difference equations:

$$\begin{cases} \Delta_t[T(y)](k) = F([T(y), \eta])(k) \\ \Delta_t[\eta](k) = G([T(y), \eta])(k) \end{cases} \quad (2.4)$$

with F and G as the following:

$$\begin{aligned} F([T(y), \eta]) &= Qs(y)(1 - \alpha(\eta)(y)) - (B + C)T(y) - A + C \int_0^1 T(y)dy \\ G([T(y), \eta]) &= \epsilon(T(\eta) - T_c) \end{aligned}$$

2.3. Equilibria and Animations. When the ice line η is fixed, the *temperature profile equilibrium, with ice line at η* is:

$$T^*(\eta)(y) = \frac{Q \cdot s(y) \cdot (1 - a(\eta)(y)) - A + C \int_0^1 T^*(\eta)(y)dy}{B + C}$$

We call the set of the Lipschitz continuous functions $\mathbb{T}^* := \{T^*(\eta) : \eta \in [0, 1]\}$ the *local equilibrium* set. This set is the steady state of the first equations in (2.4). Notice that, without a mechanism specifying the movement of the iceline, the temperature profile will dissipate to the local temperature profile $T^*(\eta)(y)$ at the initial ice line η . The following figures simulate the evolutions of the temperature profile $T_0(y) = 14 - 54y^2$ following only the first equation of (2.4), with the ice lines fixed at

$\eta_0 = 0.1, 0.3, 1$.

Note: on the electronic copy, the following figures are the initial temperature profile and the initial ice line, and the local equilibria at the respective ice lines. Click anywhere on the left figures to start the animations following Budyko's equations. Similar animations can also be found on the web through: <http://math.arizona.edu/~ewidiasih/index.html/ewidiasih/Research.html>

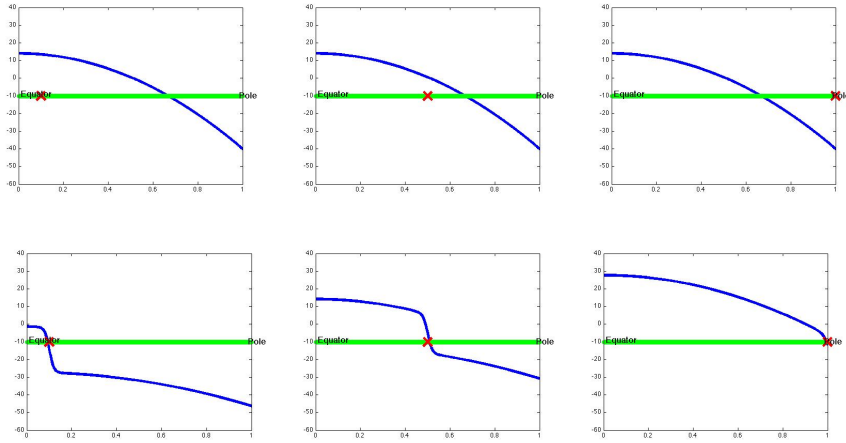


Fig. 2.1: The horizontal axes are the domain of the temperature profile T . The solid blue curve is the temperature profile, and the big red 'X' represents the location of the ice line. The figures on the left are the initial temperature profile, $T(y) = 14 - 54y^2$ and ice lines at $\eta = 0.1, 0.5$ and 1 , and the figure on the right is the steady state temperature profile $T^*(\eta)(y)$ at the respective ice lines. Clicking on the top figures will start the animation.

The next animations track the dynamics using the proposed system with parameter $\epsilon = 0.025$ and with the same starting temperature profile and ice lines as the previous series of figures. It should be noted here that we choose this value of ϵ , not based on the physical considerations, but rather to create a reasonable animations run.

Notice that the initial temperature profile first evolves to a temperature profile similar to the local equilibrium temperature profile, one with a large slope at some ice line η close to the initial ice line, and eventually they together move toward an equilibrium. This suggests the existence of an invariant manifold in the phase space (T, η) . We can directly compute the global equilibria for the proposed system by setting the ice line temperature of the local equilibria to the critical temperature and they are:

$$(T^*(\eta_1)(y), \eta_1) \text{ where } \eta_1 \cong 0.225, \text{ and } (T^*(\eta)(y), \eta_2) \text{ where } \eta_2 \cong 0.962$$

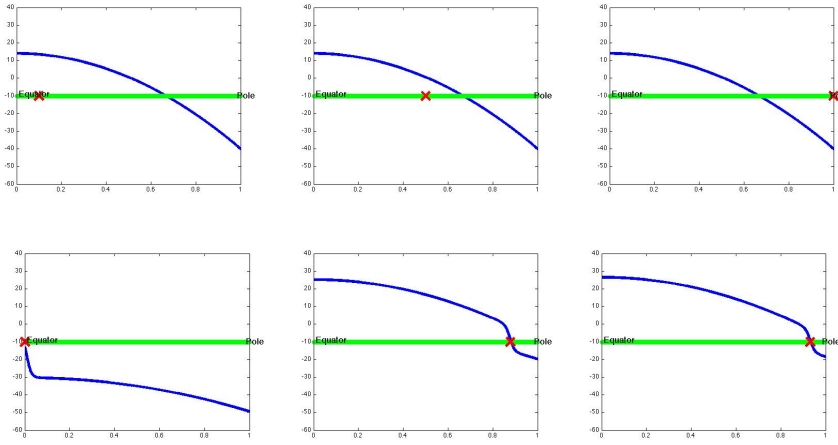


Fig. 2.2: The figures on the left are the initial temperature profile, $T(y) = 14 - 54y^2$ with ice lines at $\eta_0 = 0.1, 0.5$ and 1 , and the figure on the right is the steady state temperature profile $T^*(\eta)(y)$, with the ice lines at the equator or $\eta = 0$, $\eta \cong 0.962$. As before, clicking on the top figures will start the animations.

3. Analysis of Dynamic Iceline Budyko's Model . Our goal is to explain the dynamics of the temperature profiles $T(y)$, coupled with the ice line η in the simulation above. The main result is the existence of a one dimensional center stable manifold. The idea of an attracting invariant manifold in a fast slow system has been explored and developed by many including J. Carr [18], S. Wiggins [17], Fenichel [20], and Vanderbauwhede [19] to name a few.

First, we argue that it is reasonable to assume that the albedo functions has bounded local variation, that is, for each fixed ice line η , the albedo function $\alpha(\eta)$ has a bounded Lipschitz constant. Let M be this bound, ie. for any real values x and z ,

$$|\alpha(\eta)(x) - \alpha(\eta)(z)| < M|x - z|.$$

When the ice line is at the equator, we have an ice covered planet, otherwise, when it is at the pole, our planet is ice free. The forcings in Budyko's model as explained in section 2 is fixed to the current climate which allow for a small ice cap as an equilibrium. As mentioned in the introduction, an important question in light of the current global climate issue is:

"What does the model suggest if this equilibrium is perturbed so that the planet suddenly becomes ice free?"

To be able to describe the dynamics of the ice line η at the end points, we will embed the ice line interval and the domain of the temperature profile in the real line. We will also embed the vector field in such a way that preserves the dynamics in the unit interval including its end points. This embedding allows the dynamical analysis of the polar and the equatorial ice line and is a mathematical convenience for showing the existence of an inertial manifold.

We thus consider the following system similar to (2.4):

$$\begin{pmatrix} \Delta_t[T(y)] \\ \Delta_t[\eta] \end{pmatrix} := H([T(y), \eta]) = \begin{pmatrix} F([T(y), \eta]) \\ G([T(y), \eta]) \end{pmatrix} \quad (3.1)$$

where we define:

$$F([T(y), \eta]) :=$$

$$\begin{cases} Q \cdot s(0) \cdot [1 - a(\eta)(0)] - [A + BT(y)] + C[\int_0^1 T(y)dy - T(y)], & \text{when } y < 0 \\ Q \cdot s(y) \cdot [1 - a(\eta)(y)] - [A + BT(y)] + C[\int_0^1 T(y)dy - T(y)], & \text{when } 0 \leq y \leq 1 \\ Q \cdot s(1) \cdot [1 - a(\eta)(1)] - [A + BT(y)] + C[\int_0^1 T(y)dy - T(y)], & \text{when } 1 < y \end{cases}$$

and

$$G([T(y), \eta]) := \epsilon[T(\eta) - T_c]$$

Notice here that for a small ϵ , the vector field H separates the dynamics into 2 time scales: the fast dynamics for the temperature profiles, and the slow time scale for the ice line evolution.

Also, observe that in the extended version, any zone y outside of the unit interval has the same solar forcing as the closest endpoint. The equilibria of the extended

version are similar to the original version. The local equilibria for the temperature profile with ice line at η is the following Lipschitz continuous function:

$$T^*(\eta)(y) = \begin{cases} \frac{Q \cdot s(0) \cdot (1 - a(\eta)(0)) - A + C \int_0^1 T^*(\eta)(y)}{B + C}, & \text{when } y < 0 \\ \frac{Q \cdot s(y) \cdot (1 - a(\eta)(y)) - A + C \int_0^1 T^*(\eta)(y)}{B + C}, & \text{when } 0 \leq y \leq 1 \\ \frac{Q \cdot s(1) \cdot (1 - a(\eta)(1)) - A + C \int_0^1 T^*(\eta)(y)}{B + C}, & \text{when } 1 < y \end{cases}$$

3.1. Results. The interval of the ice line is \mathbb{R} , and we take the function space for the temperature profiles $T(y)$ to be the Banach space

$$\mathcal{B} = \{T : \mathbb{R} \rightarrow \mathbb{R} : T \text{ is bounded and continuous, with bounded Lipschitz constant}\}$$

with the norm $\|T\|_{\mathcal{B}} = \|T\|_{\infty}$

3.1.1. Inertial Manifold. We now consider the following shift operator to Dynamic Iceline Budyko's Model (3.1) in \mathcal{B} :

$$m_t([T, \eta](k)) := \begin{cases} T(k+t) = T(k) + t \cdot F([T, \eta](k)) \\ \eta(k) = \eta(k) + t \cdot G([T, \eta](k)) \end{cases}$$

We obtained the following results:

THEOREM 3.1. *For an ϵ small, there exists an attracting invariant manifold for the shift operator of Budyko's equation (3.1), that is,*

1. *There exists a Lipschitz continuous map*

$$\Phi^* : \mathbb{R} \rightarrow \mathcal{B}$$

2. *There exists a closed set $\mathcal{D} \subset \mathcal{B}$, such that for any $(T_0, \eta_0) \in \mathcal{D} \times \mathbb{R}$, the distance $\text{dist}[m_t([T_0, \eta_0])(k), (\Phi^*(\eta), \eta)]$ decreases exponentially as k increases.*

Let Φ^* denote this invariant manifold. As a corollary to the theorem (3.1), for $\eta \in [0, 1]$ we can compute the distance of the invariant manifold $\Phi^*(\eta)$ to the local equilibria manifold T^* and more importantly, we can describe the asymptotic dynamics of the ice line. The following graph is the graph of ice line temperature of the local equilibrium in the extended version, $T^*(\eta)(\eta)$ over the interval $[0, 1]$.

Many authors uses the critical temperature $T_c = -10$. The dynamics of the ice line is determined by the sign of the difference between the temperature at the ice line and the critical temperature $T(\eta) - T_c$, which is a one dimensional function of the ice line η . To determine the dynamics of the ice line at the steady state temperature profile, we graph $T^*(\eta)(\eta) - T_c$ over the interval $[0, 1]$. The function $T^*(\eta)(\eta) - T_c$ has two roots in the unit interval, as in the previous section, let the root closest to the equator, $y = 0$ be η_1 and the other root be η_2 .

We conclude from Theorem 3.1 that the dynamics reduces to 1 dimension. This fact is illustrated in Figure (3.1b). Here, we observe that when the temperature profile reaches a steady state, any ice line $\eta \in (\eta_1, \eta_2)$ moves to the right because the ice line temperature $T^*(\eta)(\eta)$ exceeds the critical temperature. On the other hand, when $\eta < \eta_1$, and $\eta > \eta_2$, the ice line temperature $T^*(\eta)(\eta)$ falls below the critical temperature, and therefore the ice line η moves to the left. In particular, we note that when the ice line is at the pole, that is when $\eta = 1$, then this ice line advances toward the equator.

COROLLARY 3.2. *In the unit interval, the invariant manifold Φ^* is within $O(\epsilon)$ of the local equilibrium set T^* . Therefore, for a small ϵ , the ice free planet is unstable.*

As an example, for a small ϵ , the equilibrium iceline temperature $\Phi^*(\eta)(\eta)$ is within $1^\circ C$ of $T^*(\eta)(\eta)$, the ice free earth is unstable.

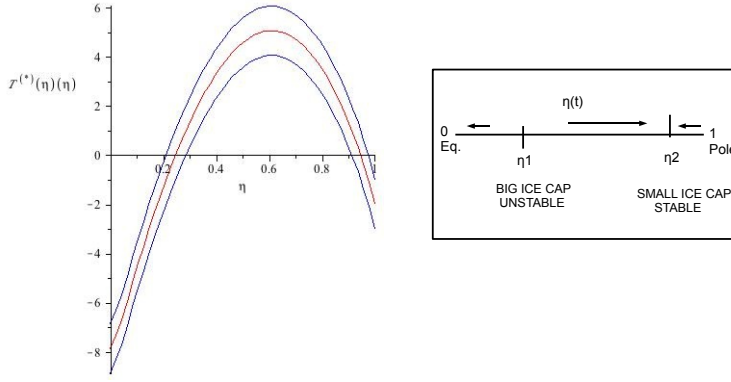


Fig. 3.1: Figure on the left illustrates $\Phi^*(\eta)(\eta)$ within $1^\circ C$ of $T^*(\eta)(\eta)$. Figure on the right is the reduced 1-D dynamics

In light of the current global climate discussion, this result suggests an optimistic view. Suppose that the planet's temperature somehow rises quickly that the ice line moves to the pole and the small ice cap disappears. If the climate parameters, ie. the parameters A, B, C are kept the same, then the temperature profile will dissipate toward that on the invariant manifold, so that the ice line temperature falls below the critical temperature, and the ice line will start moving toward the stable equilibrium, the small ice cap.

We obtain the following diagram, which is a bifurcation diagram of the solar radiation parameter Q against the ice line η in the equilibrium state. The dash blue line denotes unstable regime, and the solid line is stable, while the vertical green line is the current solar radiation used, $Q = 343$. We see that the ice free state ie. ice line at the pole or $\eta = 1$, belongs to the unstable regime when the solar radiation Q is near the current value, 343 watts per meter squared.

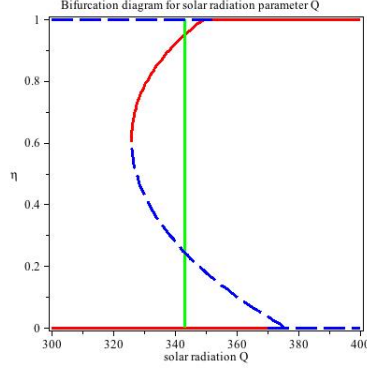


Fig. 3.2: Bifurcation diagram for Q-the solar input

4. Technical Details. This section present the proof of the existence of the 1-D center stable manifold. Readers with little interest in the mathematical treatment may wish to skip this section and proceed to the discussions on some future directions and conclusion in the next section.

We consider the extended version of the Dynamic Iceline Budyko's EBM (3.1)

4.1. A Function Space for Temperature Profiles. We take the space for the the temperature profiles to be the Banach space:

$$\mathcal{B} := \{T : \mathbb{R} \rightarrow \mathbb{R} : T \text{ is bounded, continuous with bounded Lipschits constant}\}$$

with the norm sup norm $\|T\|_\infty$

4.2. Equilibria. LEMMA 4.1. *Two Equilibria*

The system of the differential equations (3.1) has two equilibria.

Proof. First, we fix η and set $F(T, \eta)$ to zero and solve for T , in which $T = T(\eta)(y)$, a function depending on η and y . Let $T^*(\eta)(y)$ denote this solution.

$$T^*(\eta)(y) = \begin{cases} \frac{Q \cdot s(0) \cdot (1 - a(\eta)(0) - A + C \int_0^1 T^*(\eta)(y))}{B + C}, & \text{when } y < 0 \\ \frac{Q \cdot s(y) \cdot (1 - a(\eta)(y) - A + C \int_0^1 T^*(\eta)(y))}{B + C}, & \text{when } 0 \leq y \leq 1 \\ \frac{Q \cdot s(1) \cdot (1 - a(\eta)(1) - A + C \int_0^1 T^*(\eta)(y))}{B + C}, & \text{when } 1 < y \end{cases}$$

Let $g(\eta) := \int_0^1 Q \cdot s(y)(1 - a(y, \eta))dy$, then $\int_0^1 T^*(\eta)(y)dy = \frac{g(\eta) - A}{B}$, and substituting this to $T^*(\eta)$ we get an expression for $T^*(\eta)$ which only depends on η and y .

$$T^*(\eta)(y) = \begin{cases} \frac{Q \cdot s(0) \cdot (1 - a(\eta)(0)) - A + \frac{C}{B}(g(\eta) - A)}{B + C}, & \text{when } y < 0 \\ \frac{Q \cdot s(y) \cdot (1 - a(\eta)(y)) - A + \frac{C}{B}(g(\eta) - A)}{B + C}, & \text{when } 0 \leq y \leq 1 \\ \frac{Q \cdot s(1) \cdot (1 - a(\eta)(1)) - A + \frac{C}{B}(g(\eta) - A)}{B + C}, & \text{when } 1 < y \end{cases}$$

Then we use the second equation to find the ice edge(s) equilibria, that is, we set $T^*(\eta)(\eta) - T_c$ to zero, and solve for η . Let $h(\eta)$ denote the function $T^*(\eta)(\eta) - T_c$. We can eliminate the case that $\eta < 0$ or $\eta > 1$, since the temperature profiles for these ice lines do not intersect the critical temperature $T_c = -10$.

As mentioned in the introduction, we also assume that the parameter of the albedo function's maximum slope, ie. the parameter M is large, eg. $M > 10$. And we need the following computational lemma:

LEMMA 4.2. *For $M > 10$, $T^*(\eta)(\eta) - T_c$ has two roots on the unit interval.*

Let η_1 and η_2 be these roots, and suppose that $\eta_1 < \eta_2$. The equilibria for the above differential equations are therefore (T_1^*, η_1) and (T_2^*, η_2) , with

$$T_1^*(y) := T^*(\eta_1)(y) = \frac{Q \cdot (1 - a(\eta_1)(y)) \cdot s(y) + 1.6g(\eta_1) - 2.6A}{B + C}$$

$$T_2^*(y) := T^*(\eta_2)(y) = \frac{Q \cdot (1 - a(\eta_2)(y)) \cdot s(y) + 1.6g(\eta_2) - 2.6A}{B + C}$$

□

4.3. Inertial Manifold. By using Hadamard's graph transform method we will show that in an appropriate function space, for ϵ small, Dynamic Iceline Budyko's Model has an inertial manifold. The main theorem that we will prove in this section is the following:

THEOREM 4.3. *Existence of Local Inertial Manifold*

For ϵ small, there exists a locally attracting invariant manifold for the shift operator of Budyko's Energy Balance Model.

4.4. 2. The space of graphs.. For each ice line $\eta \in \mathbb{R}$, we consider $(\eta, \Phi(\eta))$, where we take the space of the graphs Φ as the space \mathcal{G} of the bounded continuous functions, $BC^0(\mathbb{R}, \mathcal{B})$ with norm $\|\Phi\|_{\mathcal{G}} = \sup_{\eta \in \mathbb{R}} \|\Phi(\eta)\|_{\mathcal{B}}$. We will need the following lemma:

LEMMA 4.4.

The set \mathcal{G} with the norm $\|\cdot\|_{\mathcal{G}}$ is a Banach space.

LEMMA 4.5.

For each fixed L , the set

$$\mathcal{G}_L = \{\Phi \in \mathcal{G} : \forall \zeta, \eta \in \mathbb{R}, \|\Phi(\zeta) - \Phi(\eta)\|_{\mathcal{B}} \leq L|\zeta - \eta|\}$$

is a closed set in the Banach space \mathcal{G} .

4.5. The albedo function $a(\eta)(y)$ as a graph. Given the location of the ice line η , the albedo function of the planet is the function $a(\eta)(y) = .47 + .15 \cdot \tanh(M \cdot (y - \eta))$. The parameter M is the steepest slope of the function, which occur on the ice line η . Furthermore, the albedo function a as a function of η is C^1 , with the upper bound for the first derivative being $.15M$, therefore, $\sup_{\eta, \zeta \in \mathbb{R}} \sup_{y \in \mathbb{R}} \frac{|a(\eta)(y) - a(\zeta)(y)|}{|\eta - \zeta|} < .15M$.

Also, for any η and y , $0.32 \leq a(\eta)(y) \leq 0.62$. Therefore, if $L \geq \max\{.62, .15M\}$, then the albedo function a is a member of \mathcal{G}_L . We will use this bound as the Lipschitz bound of the space of graphs. To be precise, we define:

DEFINITION 4.6. *Lipschitz bound, L*

The Lipschitz Bound for the space of graphs is a number L such that

$$L \geq \sup_{\eta, \zeta \in \mathbb{R}} \frac{|a(\eta) - a(\zeta)|}{|\eta - \zeta|}$$

In particular, $L \geq \max\{.62, .15m\}$.

DEFINITION 4.7. *Temperature profile bound, r*

The bound for the space of temperatures is the number r such that

$$r \geq \sup_{y \in [0, 1]} |Q \cdot s(y)|$$

We now consider the the action of Dynamics Iceline Budyko's Model over the set $\mathcal{G}_L \cap B(0, r)$ in the space of graphs, where $B(0, r) = \{\Phi \in \mathcal{G} : \|\Phi\| \leq r\}$.

4.6. The Shift Operator of Budyko's Model as a Graph Transform. We will define m a transformation that extends the vector field (F, G) so that its action is defined for all ice boundaries in the real line and for all temperature profiles in \mathcal{B} . But, first, we need a lemma:

LEMMA 4.8. *If $\epsilon < \frac{1}{L+r}$, then for any fixed $t < 1$, and for each $\eta \in \mathbb{R}$ and each $\Phi \in \mathcal{G}_L$ such that $\|\Phi\|_{\mathcal{G}} < r$, there exists a $\xi \in \mathbb{R}$ such that*

$$\eta = \xi + \epsilon \cdot t \cdot (\Phi(\xi)(\xi) - T_c)$$

Proof. We will show that given $\Phi \in \mathcal{G}_L$, there exists a unique $k = k_{\Phi} \in BC^0(\mathbb{R})$ such that for any given η , $\eta = (\eta + k(\eta)) + \epsilon(\Phi(\eta + k(\eta))(\eta + k(\eta)) - T_c)$.

We define a transformation T on $BC^0(\mathbb{R})$ by

$$(Tk)(\eta) = \epsilon t [T_c - \Phi(\eta + k(\eta))(\eta + k(\eta))]$$

Clearly, $\|Tk\| < \infty$. We will now show that T is a contraction mapping on the Banach space $BC^0(\mathbb{R})$. Indeed, for any two bounded continuous functions k_1 and k_2 with sup norm less than r , we have that:

$$\begin{aligned} & ||(Tk_1)(\eta) - (Tk_2)(\eta)|| \\ &= \epsilon t |\Phi(\eta + k_1(\eta))(\eta + k_1(\eta)) - \Phi(\eta + k_2(\eta))(\eta + k_2(\eta))| \end{aligned} \quad (4.1)$$

$$\leq \epsilon t [|\Phi(\eta + k_1(\eta))(\eta + k_1(\eta)) - \Phi(\eta + k_2(\eta))(\eta + k_1(\eta))| \quad (4.2)$$

$$+ |\Phi(\eta + k_2(\eta))(\eta + k_1(\eta)) - \Phi(\eta + k_2(\eta))(\eta + k_2(\eta))|] \quad (4.3)$$

$$\leq \epsilon t [||\Phi(\eta + k_1(\eta)) - \Phi(\eta + k_2(\eta))||_{\mathcal{B}} + ||\Phi(\eta + k_2(\eta))||_{\mathcal{B}} \cdot |k_1(\eta) - k_2(\eta)|] \quad (4.4)$$

$$\leq \epsilon t (L + r) |k_1(\eta) - k_2(\eta)| < \rho |k_1(\eta) - k_2(\eta)|. \quad (4.5)$$

with the number $\rho = \epsilon t (L + r) < 1$.

Therefore, the transformation T is a contraction on a Banach space, and by the Banach Fixed Point Theorem, there exists a unique fixed point. Let k be the fixed point, that is,

$$k(\eta) = k_{\Phi}(\eta) = \epsilon(T_c - \Phi(\eta + k(\eta))).$$

Therefore,

$$\eta + k(\eta) = \eta + \epsilon t (T_c - \Phi(\eta + k(\eta))(\eta + k(\eta))).$$

Letting $\xi = \eta + k(\eta)$ finishes the proof. \square

DEFINITION 4.9. *Graph Transform using Budyko's EBM*
Given a graph $\Phi \in \mathcal{G}_L$, we define Budyko Graph Transform m as:

$$m(\Phi) = m(\Phi)(\eta) = \Phi(\xi) + t \cdot F(\Phi, \xi)$$

with ξ as in Lemma (4.8).

To prove the existence of the inertial manifold for Dynamic Iceline Budyko's Model, we first show that m is a contraction on $\mathcal{G}_L \cap B(0, r)$.

LEMMA 4.10. *As denoted above, let L be the Lipschitz bounds, r be the temperature profiles bounds, and B be the constant from the re-emission term of DIBM. There exists an $\epsilon = \epsilon(L, r, B) > 0$ small, such that for any fixed time step t , $0 < t < \frac{1}{B+C}$, the map m is a contraction on $\mathcal{G}_L \cap \overline{B(0, r)}$.*

Proof.

We will show that for any fixed time step $t < \frac{1}{B+C}$, there exists a real number $\rho = \rho(t) < 1$ such that given Φ and Γ in $\mathcal{G}_L \cap \overline{B(0, r)}$, then $||m(\Phi) - m(\Gamma)|| < \rho ||\Phi - \Gamma||$.

By Lemma (4.8), there exists ξ and ζ such that $\eta = \xi + \epsilon \cdot t(\Phi(\xi)(\xi) - T_c)$, and $\eta = \zeta + \epsilon \cdot t(\Gamma(\zeta)(\zeta) - T_c)$. First we compare the ice boundaries $|\xi - \zeta|$ to their temperatures $|\Phi(\xi)(\xi) - \Gamma(\zeta)(\zeta)|$.

$$\begin{aligned} |\xi - \zeta| &\leq \epsilon \cdot t |\Phi(\xi)(\xi) - \Gamma(\zeta)(\zeta)| \\ &\leq \epsilon \cdot t [|\Phi(\xi)(\xi) - \Gamma(\xi)(\xi)| + |\Gamma(\xi)(\xi) - \Gamma(\xi)(\zeta)| + |\Gamma(\xi)(\zeta) - \Gamma(\zeta)(\zeta)|] \\ &\leq \epsilon \cdot t ||\Phi - \Gamma|| \quad \text{by the definition} \\ &\quad + r \epsilon t |\xi - \zeta| \quad \text{since for each } \xi, \Gamma(\xi) \in \mathcal{B} \text{ and } ||\Gamma|| < r \\ &\quad + L \epsilon t |\xi - \zeta| \quad \text{since } \Gamma \in \mathcal{G}_L \\ &\leq \epsilon \cdot t ||\Phi - \Gamma|| + \epsilon(L + r) |\xi - \zeta| \end{aligned}$$

Solving for $|\xi - \zeta|$ we get the inequality:

$$|\xi - \zeta| \leq \frac{t \cdot \epsilon}{1 - (L + r)\epsilon} \|\Phi - \Gamma\| \quad (4.6)$$

Let

$$\epsilon \leq \frac{B}{2(Lr + L + r)} \quad (4.7)$$

and define:

$$\delta_1 := \frac{L \cdot \epsilon}{1 - (L + r)\epsilon} \text{ and} \quad (4.8)$$

$$\delta_2 := \frac{L \cdot r \cdot \epsilon}{1 - (L + r)\epsilon} \quad (4.9)$$

Then

$$\delta_1 < \delta_2 < \frac{B}{2}. \quad (4.10)$$

We now estimate the graph transform map m :

$$\begin{aligned} & |m(\Phi)(\eta)(y) - m(\Gamma)(\eta)(y)| \\ &= |\Phi(\xi)(y) + t \cdot [Q \cdot s(y) \cdot (1 - \alpha(\xi)(y)) - (B + C)\Phi(\xi)(y) + C\overline{\Phi(\xi)} - A] \\ &\quad - \Gamma(\xi)(y) - t \cdot [Q \cdot s(y) \cdot (1 - \alpha(\xi)(y)) - (B + C)\Gamma(\xi)(y) + C\overline{\Gamma(\xi)} - A] \\ &\quad + \Gamma(\xi)(y) + t \cdot [Q \cdot s(y) \cdot (1 - \alpha(\xi)(y)) - (B + C)\Gamma(\xi)(y) + C\overline{\Gamma(\xi)} - A] \\ &\quad - \Gamma(\zeta)(y) - t \cdot [Q \cdot s(y) \cdot (1 - \alpha(\zeta)(y)) - (B + C)\Gamma(\zeta)(y) + C\overline{\Gamma(\zeta)} - A]| \end{aligned}$$

Since $t < \frac{1}{B+C}$, and therefore $1 - t(B+C) > 0$, then the estimate above continues as

$$\begin{aligned} &\leq [1 - t(B + C)]|\Phi(\xi)(y) - \Gamma(\xi)(y)| + t \cdot C|\overline{\Phi(\xi)} - \overline{\Gamma(\xi)}| \\ &\quad + [1 - t(B + C)]|\Gamma(\xi)(y) - \Gamma(\zeta)(y)| + t \cdot C|\overline{\Gamma(\xi)} - \overline{\Gamma(\zeta)}| \\ &\quad + |Q \cdot s(y) \cdot (\alpha(\xi)(y) - \alpha(\zeta)(y))| \end{aligned}$$

Using inequality (4.6) we estimate the third through the fifth terms of the above inequality:

$$\begin{aligned} &\leq [1 - t(B + C) + tC]\|\Phi - \Gamma\| \\ &\quad + [1 - t(B + C)](t\delta_1)\|\Phi - \Gamma\| + (tC)(t\delta_1)\|\Phi - \Gamma\| \\ &\quad + t\delta_2\|\Phi - \Gamma\| \\ &\leq (1 - tB + t[(1 - tB)\delta_1 + \delta_2])\|\Phi - \Gamma\| \end{aligned}$$

By the choice of ϵ in (4.7) above, we get the inequality (4.10), and so the inequality above continues as:

$$\begin{aligned} &< ((1 - tB) + t(\delta_2 + \delta_2)) \|\Phi - \Gamma\| \\ &\leq 1 \cdot \|\Phi - \Gamma\| \end{aligned}$$

Therefore, for any fixed t , $0 < t < \frac{1}{B+C}$, if $\rho := ((1 - tB) + t(\delta_2 + \delta_2))$, then $0 < \rho < 1$, and we showed that:

$$\|m(\Phi) - m(\Gamma)\| < \rho \|\Phi - \Gamma\|.$$

That is, the map m is a contraction. \square

We finish the proof of the existence of invariant manifold (4.3):

Proof. For ϵ as in the previous lemma, we showed that

$$m : \mathcal{G}_L \cap B(0, r) \rightarrow \mathcal{G}_L \cap B(0, r)$$

is a contraction in a closed set of a Banach space. Therefore, there exists a unique fixed point Φ^* such that $m(\Phi^*) = \Phi^*$. \square

COROLLARY 4.11. *The invariant manifold Φ^* is within $O(\epsilon)$ of the equilibrium set T^**

Proof. Since $m(\Phi^*) = \Phi^*$, then the following holds:

$$\Phi^*(\eta) = \Phi^*(\xi) + tF(\Phi^*(\xi) - T^*(\xi) + T^*(\xi), \xi)$$

where $\xi = \eta + k_{\Phi^*}(\eta)$.

Using the facts that $F(T^*(\xi), \xi) = 0$ and the reverse triangle inequality we find:

$$\begin{aligned} \|\Phi^*(\eta) - \Phi^*(\xi)\|_{\mathcal{B}} &= \|t(B + C) \cdot [\Phi^*(\xi) - T^*(\xi)] - tB(\overline{\Phi^*(\xi)} - \overline{T^*(\xi)})\|_{\mathcal{B}} \\ &\geq \left| t(B + C) \cdot \|\Phi^*(\xi) - T^*(\xi)\|_{\mathcal{B}} - tB\|\overline{\Phi^*(\xi)} - \overline{T^*(\xi)}\|_{\mathcal{B}} \right| \\ &\geq tB\|\Phi^*(\xi) - T^*(\xi)\|_{\mathcal{B}} \end{aligned}$$

In the last step of the above estimate, we used the fact that $\left| \overline{\Phi^*(\xi)} - \overline{T^*(\xi)} \right| \leq \|\Phi^*(\xi) - T^*(\xi)\|_{\mathcal{B}}$.

Therefore, we arrive at the following estimate:

$$\begin{aligned} \|\Phi^*(\xi) - T^*(\xi)\| &= \frac{1}{B} \|\Phi^*(\eta) - \Phi^*(\eta + k_{\Phi^*}(\eta))\| \\ &\leq \frac{1}{B} \cdot |\Phi^*(\eta + k_{\Phi^*}(\eta))(\eta + k_{\Phi^*}(\eta)) - T_c| \\ &\leq \frac{\epsilon r}{B} \end{aligned}$$

recall that here, r is the bound on the temperature profiles. \square

COROLLARY 4.12. *For ϵ small, the ice free earth is unstable.*

Proof. Since $\|\Phi^* - T^*\| < \frac{\epsilon \cdot r}{B}$ then $|\Phi^*(\eta)(\eta) - T^*(\eta)(\eta)| < \frac{\epsilon \cdot r}{B}$ as well. So that if $\epsilon < \frac{B}{r}[T_c - T^*(1)(1)]$, then $\Phi^*(1)(1) < T_c$, and so, ice will form at the pole and advances toward the small ice cap equilibrium. See figure (??) for the graph of the iceline local equilibrium temperature $T^*(\eta)(\eta)$.

Therefore, if $\eta(0) = 1$, as $t \rightarrow \infty$, $\eta(t)$, the ice line, advances equatorward, and the planet evolves toward a non ice free earth. \square

5. Concluding Discussion.

5.1. Future Directions. There are several directions that one can explore based on this model. One immediate improvement is to compute the invariant manifold explicitly as done by Foias, Sell and Titi [21]. Another immediate improvement on this model is an extension to the southern hemisphere with two ice line and a non symmetric transport. Another direction is to explore how the change in the greenhouse gas components, which is the term $A + BT(y)$, affects the radiative forcings. A work by Andrew Hogg [15] relates the evolution of temperature with that of carbondioxide as a response to the solar input variations caused by the Milankovitch cycle. We are interested in the possibility of coupling Budyko's model with the Hogg's model to understand the glacial cycles in the quaternary period. While North explores a similar model with only a diffusion transport [2], [3], [4], the model discussed in this paper could be improved by including some diffusion and averaging in the transport term. Such inclusion necessitates the consideration for the planet's heat capacity and a further explanation of the parameter ϵ .

5.2. Conclusion. We have shown in this paper the existence of a center stable manifold in an energy balance model with ice albedo feedback, featuring a dynamic iceline. The existence of such invariant manifold explains the numerical experiments presented as animations and allows for qualitative analysis of the small icecap stability.

REFERENCES

- [1] I. EISENMAN AND J. WETTLAUFER, *Nonlinear threshold behavior during the loss of Arctic sea ice*, in Proceeding of National Academic of Science, vol. 106, Number 1, January 6, 2009.
- [2] G. R. NORTH, *Theory of Energy Balance Climate Models*, in Journal of the Atmospheric Sciences, Volume 36, Issue 11, 1975, online.
- [3] R. CAHALAN AND G. R. NORTH, *A Stability Theorem for Energy Balance Climate Models*, in Journal of the Atmospheric Sciences, Volume 32, Issue 11, 1979, online.
- [4] G. R. NORTH, *Small Ice Cap Instability in Diffusive Climate Models*, Journal of Atmospheric Science, Volume 41, Issue 23, 1984.
- [5] G. R. NORTH, *Multiple solutions in energy balance climate models*, Palaeogeography, Palaeoclimatology, Palaeoecology (Global and Planetary Change Section) 80: 225-235 Elsevier Science Publishers B.V. Amsterdam, 1990.
- [6] C.E. GRAVES, W.H. LEE, G. R. NORTH, *New Parametrizations and Sensitivities for Simple Climate Models*, Journal of Geophysical Research, Volume 98, No D3: 5025-5036, March 20, 1993.
- [7] M.I. BUDYKO, *The effect of solar radiation on the climate of the earth*, Tellus, 21, 611-619, (1969).
- [8] K.K. TUNG, *Topics on Mathematical Modeling*, Princeton University Press, (2007).
- [9] P. A. SIMMONS AND D. H. GRIFFEL, *Continuous versus discontinuous albedo representations in a simple diffusive climate model*, in Climate Dynamics, Volume 3, Number 1, pp. 35-39, 1988.
- [10] P.G. DRAZIN AND D.H. GRIFFEL, *On Diffusive Climatological Model*, in Journal Atmospheric Science, 38:2237-2232, 1981.
- [11] W.J. Merryfield, M.M. Holland, and A.H. Monahan Multiple equilibria and abrupt transitions in Arctic summer sea ice extent, In Arctic Sea Ice Decline: Observations, Projections, Mechanisms, and Implications, Geophys. Monogr. Ser., 180, pp. 151-174, AGU, Washington, D. C., 2008.
- [12] WILLIAM SELLERS, *A climate model based on the energy balance of the earth-atmosphere system*, J. Applied Metereology. 8: 392-400, 1969.
- [13] DIRK NOTZ, *Tipping Elements in Earth Systems Special Feature: The future of ice sheets and sea ice: Between reversible retreat and unstoppable loss*, in Proceeding of National Academic of Science, vol. 106, Number 49, December 8, 2009.
- [14] C.E.P BROOKS *The problem of warm polar climates*, Q.J.R. of Metereological Society, 51:868-874, 1925.
- [15] A. MCC. HOGG *Glacial cycles and carbon dioxide: A conceptual model*, GEOPHYSICAL RESEARCH LETTERS, VOL. 35, L01701, 5 PP., 2008.
- [16] H. AMANN, *Ordinary Differential Equations: an introduction to nonlinear analysis*, de Gruyter Studies in Mathematics, NY, NY, 1990.
- [17] STEPHEN WIGGINS, *Normally Hyperbolic Invariant Manifolds in Dynamical Systems*, Springer-Verlag, NY, NY, 1994.
- [18] JACK CARR, *Applications of Centre Manifold Theory*, Springer-Verlag, NY, NY, 1981.
- [19] A. VANDERBAUWHEDE *Center Manifolds, Normal Forms and Elementary Bifurcations*, In Dynamics Reported, Vol. 2. Wiley. 1989.
- [20] N. FENICHEL *Geometric singular perturbation theory*; JDE 31, 53-98.1979.
- [21] CIPRIAN FOIAS, GEORGE R. SELL AND EDRISS S. TITI *Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations*, Journal of Dynamics and Differential Equations, Springer Netherlands. Volume 1, Number 2, 1989.
- [22] R. MCGEHEE Personal Communication, 2010.